[7] One could also take for J' an *injective* extension E of J. For any J there exists such an E, but the proof of this fact is rather complicated, and it is much simpler to take $J' = \Phi(R, J)$. It is easily seen that $\Phi(R, J)$ is injective in a *weaker* sense, which is sufficient for the cohomology theory of groups.

[8] Steenrod, N. E., *Ann. Math.*, **44**, 610–627 (1943).

[9] Eckmann, B., Cohomology of groups and transfer, to appear in *Ann. Math.*

[10] Hopf, H., *Comment. Math. Helv.*, **16**, 81–100 (1943), and Freudenthal, H., *Ibid.*, **17**, 1–38 (1944).

[11] Specker, E., *Ibid.*, **23**, 303–332 (1949), theorem IV (§5).

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*MINIMAX THEOREMS**

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Various generalizations of von Neumann's minimax theorem[1] have been given by several authors (J. Ville,[2] A. Wald,[2] S. Karlin,[2] H. Kneser,[3] K. Fan[4]). In all these theorems, the structure of linear spaces is always present. This note contains some new minimax theorems involving no linear space.

1. Let f be a real-valued function defined on the product set $X \times Y$ of two arbitrary sets X, Y (not necessarily topologized). f is said to be *convex on* X, if for any two elements $x_1, x_2 \in X$ and two numbers $\xi_1 \geqq 0$, $\xi_2 \geqq 0$ with $\xi_1 + \xi_2 = 1$, there exists an element $x_0 \in X$ such that $f(x_0, y) \leqq \xi_1 f(x_1, y) + \xi_2 f(x_2, y)$ for all $y \in Y$. Similarly f is said to be *concave on* Y, if for any two elements $y_1, y_2 \in Y$ and two numbers $\eta_1 \geqq 0$, $\eta_2 \geqq 0$ with $\eta_1 + \eta_2 = 1$, there exists an $y_0 \in Y$ such that $f(x, y_0) \geqq \eta_1 f(x, y_1) + \eta_2 f(x, y_2)$ for all $x \in X$.

THEOREM 1. *Let X, Y be two compact Hausdorff spaces and f a real-valued function defined on $X \times Y$. Suppose that, for every $y \in Y$, $f(x, y)$ is lower semi-continuous (l.s.c.) on X; and for every $x \in X$, $f(x, y)$ is upper semi-continuous (u.s.c.) on Y. Then:*

(i) *The equality*

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y) \tag{1}$$

holds, if and only if the following condition is satisfied: For any two finite sets $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\{y_1, y_2, \ldots, y_m\} \subset Y$, there exist $x_0 \in X$ and $y_0 \in Y$ such that

$$f(x_0, y_k) \leqq f(x_i, y_0). \qquad (1 \leqq i \leqq n,\ 1 \leqq k \leqq m) \tag{2}$$

(ii) *In particular, if f is convex on X and concave on Y, then (1) holds.*

Proof: Observe first that, regardless of the condition stated in (i), the expressions on both sides of (1) are meaningful. In fact, for each $x \in X$, $f(x, y)$ is u.s.c. on the compact space Y, so that $\max_{y \in Y} f(x, y)$ exists. As maximum of a family of l.s.c. functions on X, $\max_{y \in Y} f(x, y)$ is a l.s.c. function on the compact space X and therefore $\min_{x \in X} \max_{y \in Y} f(x, y)$ exists.

(i) The necessity of the condition being trivial, we only prove its sufficiency. According to this condition,

$$\min_{x \in X} \max_{1 \leqq k \leqq m} f(x, y_k) \leqq \max_{y \in Y} \min_{1 \leqq i \leqq n} f(x_i, y)$$

holds for any two finite sets $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\{y_1, y_2, \ldots, y_m\} \subset Y$. Then any real number α satisfies at least one of the two inequalities

$$\min_{x \in X} \max_{1 \leqq k \leqq m} f(x, y_k) \leqq \alpha, \qquad \max_{y \in Y} \min_{1 \leqq i \leqq n} f(x_i, y) \geqq \alpha.$$

Let $L(y; \alpha) = \{x \in X | f(x, y) \leqq \alpha\}$, $U(x; \alpha) = \{y \in Y | f(x, y) \geqq \alpha\}$, which are closed subsets of X, Y, respectively. Then for any real α and any two finite sets $\{x_1, x_2, \ldots, x_n\} \subset X$, $\{y_1, y_2, \ldots, y_m\} \subset Y$, the two intersections $\bigcap_{k=1}^{m} L(y_k; \alpha)$ and $\bigcap_{i=1}^{n} U(x_i; \alpha)$ are never both empty. As X, Y are compact, it follows that for any real α, at least one of the two intersections $\bigcap_{y \in Y} L(y; \alpha)$ and $\bigcap_{x \in X} U(x; \alpha)$ is not empty. That is, either there exists $x_0 \in X$ such that $f(x_0, y) \leqq \alpha$ for all $y \in Y$, or there exists $y_0 \in Y$ such that $f(x, y_0) \geqq \alpha$ for all $x \in X$. In other words, any real α satisfies at least one of the two inequalities

$$\min_{x \in X} \max_{y \in Y} f(x, y) \leqq \alpha, \qquad \max_{y \in Y} \min_{x \in X} f(x, y) \geqq \alpha.$$

Hence $\min_{x \in X} \max_{y \in Y} f(x, y) \leqq \max_{y \in Y} \min_{x \in X} f(x, y)$ and therefore (1).

(ii) Assume now that f is convex on X and concave on Y. In order to prove (1), it suffices to verify the condition stated in (i). Let $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\{y_1, y_2, \ldots, y_m\} \subset Y$ be given. By von Neumann's minimax theorem,[1] there exist two sets $\{\xi_1, \xi_2, \ldots, \xi_n\}$, $\{\eta_1, \eta_2, \ldots, \eta_m\}$ of non-negative numbers with $\sum_{i=1}^{n} \xi_i = 1$, $\sum_{k=1}^{m} \eta_k = 1$ such that

$$\max_{1 \leqq k \leqq m} \sum_{i=1}^{n} \xi_i f(x_i, y_k) \leqq \min_{1 \leqq i \leqq n} \sum_{k=1}^{m} \eta_k f(x_i, y_k). \tag{3}$$

Since f is convex on X and concave on Y, there exist $x_0 \in X$, $y_0 \in Y$ such that

$$f(x_0, y_k) \leqq \sum_{i=1}^{n} \xi_i f(x_i, y_k), \qquad (1 \leqq k \leqq m) \tag{4}$$

$$f(x_i, y_0) \geqq \sum_{k=1}^{m} \eta_k f(x_i, y_k). \qquad (1 \leqq i \leqq n) \tag{5}$$

Then (2) follows from (3), (4), (5).

In (ii) of Theorem 1, we have an easily applicable sufficient condition for equality (1). It can be used to provide simple proofs for minimax theorems for infinite games (for instance, the generalized Ville's minimax theorem as stated in our earlier note[4] is a special case of part (ii) of Theorem 1). Since the proof of (ii) is based on von Neumann's minimax theorem, its application in proving a minimax theorem for infinite games amounts essentially to a reduction of the latter to von Neumann's theorem for finite games.

2. Theorem 2 below generalizes Kneser's minimax theorem[3] by eliminating the structure of linear spaces. Theorem 2 also improves (ii) of Theorem 1. Our proof of Theorem 2 is a modification of Kneser's proof of his theorem.

THEOREM 2. *Let X be a compact Hausdorff space and Y an arbitrary set (not topologized). Let f be a real-valued function on $X \times Y$ such that, for every $y \in Y$, $f(x, y)$ is l.s.c. on X. If f is convex on X and concave on Y, then*

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y). \tag{6}$$

Proof: Observe first that the expressions on both sides of (6) have meaning, although their values may be $+\infty$. We divide the proof of (6) into four steps:

(i) *Let $y_0 \in Y$ be such that $X_0 = \{x \in X | f(x, y_0) \leqq 0\}$ is not empty. If we replace X by X_0, and restrict f on $X_0 \times Y$, then the hypothesis of Theorem 2 remains fulfilled.*

We need only to verify that f restricted on $X_0 \times Y$ is convex on X_0. Let $x_1, x_2 \in X_0$, and $\xi_1, \xi_2 \geqq 0$, $\xi_1 + \xi_2 = 1$ be given. By convexity of f on X, there exists $x_0 \in X$ such that

$$f(x_0, y) \leqq \xi_1 f(x_1, y) + \xi_2 f(x_2, y) \text{ for all } y \in Y. \tag{7}$$

Since $f(x_1, y_0) \leqq 0, f(x_2, y_0) \leqq 0$, the case $y = y_0$ of (7) implies $f(x_0, y_0) \leqq 0$, i.e., $x_0 \in X_0$.

(ii) *If $\{y_1, y_2\} \subset Y$ is such that*

$$\max_{k=1,2} f(x, y_k) > 0 \textit{ for all } x \in X, \tag{8}$$

then there exists $y_0 \in Y$ such that

$$f(x, y_0) > 0 \textit{ for all } x \in X \tag{9}$$

Let $X_k = \{x \in X | f(x, y_k) \leqq 0\}$, $(k = 1, 2)$, which are disjoint closed sets in X. We assume that none of them is empty (otherwise (ii) is trivial).

We have $-f(x, y_1) \geqq 0$ and $f(x, y_2) > 0$ for $x \in X_1$, so that $\dfrac{-f(x, y_1)}{f(x, y_2)}$ is u.s.c. and $\geqq 0$ on X_1. Let $x_1 \in X_1$ and $\mu_1 \geqq 0$ be such that

$$\max_{x \in X_1} \frac{-f(x, y_1)}{f(x, y_2)} = \frac{-f(x_1, y_1)}{f(x_1, y_2)} = \mu_1. \tag{10}$$

Similarly, let $x_2 \in X_2$ and $\mu_2 \geqq 0$ be such that

$$\max_{x \in X_2} \frac{-f(x, y_2)}{f(x, y_1)} = \frac{-f(x_2, y_2)}{f(x_2, y_1)} = \mu_2. \tag{11}$$

We claim that $\mu_1\mu_2 < 1$. To verify this, we may assume $\mu_1\mu_2 \neq 0$. Since $f(x_1, y_1) \leqq 0, f(x_2, y_1) > 0$, we can find $\xi_1, \xi_2 \geqq 0$ such that $\xi_1 + \xi_2 = 1$ and

$$\xi_1 f(x_1, y_1) + \xi_2 f(x_2, y_1) = 0. \tag{12}$$

f being convex on X, there exists $x_0 \in X$ such that

$$f(x_0, y) \leqq \xi_1 f(x_1, y) + \xi_2 f(x_2, y) \text{ for all } y \in Y. \tag{13}$$

From (12), (13), we have $f(x_0, y_1) \leqq 0$ and therefore, by (8), $f(x_0, y_2) > 0$, so that

$$0 < \xi_1 f(x_1, y_2) + \xi_2 f(x_2, y_2).$$

Using (10), (11) and the fact $\mu_1 > 0$, the last inequality may be written

$$\xi_1 f(x_1, y_1) + \mu_1\mu_2\xi_2 f(x_2, y_1) < 0,$$

which compared with (12) yields $\mu_1\mu_2 < 1$.

Take ν_1, ν_2 such that $\nu_1 > \mu_1$, $\nu_2 > \mu_2$, $\nu_1\nu_2 = 1$. Let

$$\eta_1 = \frac{1}{1 + \nu_1} = \frac{\nu_2}{1 + \nu_2}, \qquad \eta_2 = \frac{\nu_1}{1 + \nu_1} = \frac{1}{1 + \nu_2}.$$

Then

$$\eta_1 f(x, y_1) + \eta_2 f(x, y_2) > 0 \text{ for all } x \in X. \tag{14}$$

In fact, if x is not in $X_1 \cup X_2$, (14) is trivial. If $x \in X_1$, we have $0 \leqq f(x, y_1) + \mu_1 f(x, y_2) < f(x, y_1) + \nu_1 f(x, y_2) = (1 + \nu_1)[\eta_1 f(x, y_1) + \eta_2 f(x, y_2)]$. Similarly one verifies (14) for $x \in X_2$. Finally the existence of $y_0 \in Y$ with property (9) follows from (14) and the concavity of f on Y.

(iii) *If a finite set* $\{y_1, y_2, \ldots, y_m\} \subset Y$ *is such that*

$$\max_{1 \leqq k \leqq m} f(x, y_k) > 0 \textit{ for all } x \in X, \tag{15}$$

then there exists $y_0 \in Y$ *satisfying* (9).

We prove this by induction on m. Let $X_m = \{x \in X | f(x, y_m) \leqq 0\}$.

We assume that X_m is not empty (otherwise we take $y_0 = y_m$). By (i), we can apply our induction-assumption to f restricted on $X_m \times Y$. Since $\max_{1 \leq k \leq m-1} f(x, y_k) > 0$ for all $x \in X_m$, there exists $y_{m+1} \in Y$ such that $f(x, y_{m+1}) > 0$ for all $x \in X_m$. Then $\max_{k=m, m+1} f(x, y_k) > 0$ for all $x \in X$. By (ii), there exists $y_0 \in Y$ satisfying (9).

(iv) *For any real number α, either there exists an $x_0 \in X$ such that $f(x_0, y) \leq \alpha$ for all $y \in Y$, or there exists an $y_0 \in Y$ such that $f(x, y_0) > \alpha$ for all $x \in X$.* (Therefore the right side of (6) is not less than the left side, and consequently the two are equal.)

Suppose the first alternative is not true. Then $\bigcap_{y \in Y} L(y; \alpha)$ is empty, where $L(y; \alpha) = \{x \in X | f(x, y) \leq \alpha\}$. As X is compact, there exists a finite set $\{y_1, y_2, \ldots, y_m\} \subset Y$ such that $\bigcap_{k=1}^{m} L(y_k; \alpha)$ is empty. That is, $\max_{1 \leq k \leq m} f(x, y_k) > \alpha$ for all $x \in X$. Then an application of (iii) to the function $f - \alpha$ shows that the second alternative is true.

3. The next theorem is free of topological structures. This is made possible by generalizing von Neumann's almost periodic functions on a group:[5] A real-valued function f defined on the product set $X \times Y$ of two arbitrary sets X, Y (not topologized) is said to be *right almost periodic*, if f is bounded on $X \times Y$ and if, for any $\epsilon > 0$, there exists a finite covering $Y = \bigcup_{k=1}^{m} Y_k$ of Y such that $|f(x, y') - f(x, y'')| < \epsilon$ for all $x \in X$, whenever y', y'' belong to the same Y_k. *Left almost periodic* functions are defined similarly. However, every right almost periodic function on $X \times Y$ is also left almost periodic and vice versa.[6] Thus we simply use the term *almost periodic*.

THEOREM 3.[7] *Let f be a real-valued almost periodic function on the product set $X \times Y$ of two arbitrary sets X, Y (not topologized). Then:*

(i) *The equality*

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y) \tag{16}$$

holds, if and only if the following condition is satisfied: For any $\epsilon > 0$, any two finite sets $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\{y_1, y_2, \ldots, y_m\} \subset Y$, there exist $x_0 \in X$, $y_0 \in Y$ such that

$$f(x_0, y_k) - f(x_i, y_0) \leq \epsilon. \quad (1 \leq i \leq n, 1 \leq k \leq m). \tag{17}$$

(ii) *In particular, if f is convex on X and concave on Y, then* (16) *holds.*

Proof: We only prove the sufficiency part of (i). Consider an arbitrary $\epsilon > 0$. Let $X = \bigcup_{i=1}^{n} X_i$ be a finite covering of X with the property corresponding to ϵ required in the definition of left almost periodicity. Let $Y = \bigcup_{k=1}^{m} Y_k$ be a finite covering of Y with the property corresponding to ϵ required in the definition of right almost periodicity. Let $x_i \in X_i$ $(1 \leqq i \leqq n)$, $y_k \in Y_k$ $(1 \leqq k \leqq m)$. Then $\sup_{y \in Y} f(x, y) \leqq \max_{1 \leqq k \leqq m} f(x, y_k) + \epsilon$ holds for all $x \in X$; and $\inf_{x \in X} f(x, y) \geqq \min_{1 \leqq i \leqq n} f(x_i, y) - \epsilon$ holds for $y \in Y$.[8] Hence

$$\inf_{x \in X} \sup_{y \in Y} f(x, y) \leqq \inf_{x \in X} \max_{1 \leqq k \leqq m} f(x, y_k) + \epsilon, \tag{18}$$

$$\sup_{y \in Y} \inf_{x \in X} f(x, y) \geqq \sup_{y \in Y} \min_{1 \leqq i \leqq n} f(x_i, y) - \epsilon. \tag{19}$$

Using our condition, there exist $x_0 \in X$, $y_0 \in Y$ satisfying (17). From (17), (18), (19), we get

$$0 \leqq \inf_{x \in X} \sup_{y \in Y} f(x, y) - \sup_{y \in Y} \inf_{x \in X} f(x, y) \leqq 3\epsilon,$$

which holds for any $\epsilon > 0$. Thus (16) is proved.

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[5] Von Neumann, J., "Almost periodic functions in a group, I," *Trans. Am. Math. Soc.*, **36**, 445–492 (1934); Maak, W., "Eine neue Definition der fast-periodischen Funktionen," *Abhandl. Math. Sem. Hans. Univ.*, **11**, 240–244 (1936).

[6] Let f be right almost periodic. Given $\epsilon > 0$, let $Y = \bigcup_{k=1}^{m} Y_k$ be a finite covering of Y with the property corresponding to $\epsilon/3$ required int he definition of right almost periodicity. Let $y_k \in Y_k$ $(1 \leqq k \leqq m)$. For each k, $f(x, y_k)$ is a bounded function on X, there is a finite covering $X = \bigcup_{i=1}^{nk} X_i^{(k)}$ of X such that $|f(x', y_k) - f(x'', y_k)| < \frac{\epsilon}{3}$ whenever $x', x'' \in X_i^{(k)}$ for some i. Then the common refinement of these m finite coverings of X has the property corresponding to ϵ required in the definition of left almost periodicity.

[7] It should be said that the absence of topological structures in Theorem 3 is more apparent than real. In fact, the almost periodicity of f is a necessary and sufficient condition in order that X, Y can be made into two precompact (in the sense of Bourbaki, but not necessarily separated) uniform spaces in such a way that f is uniformly continuous on the product uniform space $X \times Y$.

[8] Here we see that the hypothesis in Theorem 3 on almost periodicity of f can be considerably weakened.